Math 122 / Problem Set 9

Written problems due Wednesday, November 30

Friday, November 18

1. Prove that $\alpha = \sqrt{3} + \sqrt{-5}$ is an algebraic number by finding an explicit polynomial of degree 4 with integer coefficients which is satisfied by α.

2. Let F be a field. Determine the units in the polynomial ring $F[x]$.

3. The *product ring* $R \times R'$ of two rings is defined to be the product set $\{(r, r') : r \in R, r' \in R'\}$ with component-wise addition and multiplication. That is, $(r, r') + (s, s') = (r+s, r'+s')$ and $(r, r')(s, s') = (rs, r's')$. Prove that $R \times R'$ is a ring.

Reading: Artin §10.1, §10.2, §10.3

Monday, November 21

4. Prove or disprove: If an ideal I contains a unit, then it is the unit ideal (= the entire ring).

5. Prove that every nonzero ideal in the ring of Gaussian integers contains a nonzero integer.

6. Describe the kernel of the following maps.
(a) $\varphi : \mathbb{R}[X, Y] \to \mathbb{R}$ defined by $\varphi(f(X, Y)) = f(0, 0)$.
(b) $\varphi : \mathbb{R}[X] \to \mathbb{C}$ defined by $\varphi(f(X)) = f(2 + i)$.

Reading: Artin §10.4, §10.7

Wednesday, November 23

7. Let I, J be ideals in a ring R such that $I + J = R$.
(a) Prove that $IJ = I \cap J$. (*Hint:* Show that $IJ \subset I \cap J$ for any ideals I, J. Then using $u \in I$, $v \in J$ such that $u + v = 1$, show that $I \cap J \subset IJ$.)
(b) Prove the *Chinese Remainder Theorem:* For any pair a, b of elements of R, there is an element x such that $x \equiv a \pmod{I}$ and $x \equiv b \pmod{J}$. (The notation $x \equiv a \pmod{I}$ means $x - a \in I$. *Hint:* Use u, v as in **(a)**.)
(c) Show that $R/IJ \simeq R/I \times R/J$.

8. Let R be a ring, and let I be an ideal of R. Let M be an ideal of R containing I, and let $\overline{M} = M/I$ be the corresponding ideal of $\overline{R} = R/I$. Prove that M is maximal if and only if $\overline{M}$ is.

9. Let $R = \mathbb{C}[x_1, \ldots, x_n]/I$ be a quotient of a polynomial ring over $\mathbb{C}$, and let M be a maximal ideal of R. Prove that $R/M \simeq \mathbb{C}$.

Reading: Artin §10.5

Monday, November 28

10. Suppose we adjoin an element α to $\mathbb{R}$ satisfying the relation $\alpha^2 = 1$. Prove that the resulting ring is isomorphic to the product ring $\mathbb{R} \times \mathbb{R}$, and find the element of $\mathbb{R} \times \mathbb{R}$ which corresponds to α.

Reading: Artin §10.6